TRANSFER FUNCTION OF A TIME-VARYING CONTROL SYSTEM

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INTRODUCTION

The study of linear time-varying systems (LTV) is an integral part of the theory of automatic control, the development of which is caused by the need to solve a number of technical problems, in particular, the design of aircraft motion control systems. In order to determine the control law (CL), which ensures the given parameters of the LTV, various variants of the mathematical apparatus have been used, for example, differential inequalities and parametric Lyapunov equations, predictive control models, differential equations with constant coefficients around a certain time.

Analysis of stability of LTV compared to stationary systems is much more complicated for several reasons. First, another formulation of the concept of stability, secondly, there is no obvious connection between the stability of the LTV and the eigenvalues of the matrix of the equations system. In addition, the result of the analysis largely depends on the state transition matrices, the possibility of determining which is obvious not always [1].

The construction of Lyapunov function (LF) for LTV is related to the solution of a scalar differential equation, which contains both improper and double integrals [2]. For scalar LTV, a method of LF construction based on the use of the integral of the system parameter with a weight function on a finite interval is proposed. Conditions are imposed on the weight function so that LF is positively defined and uniformly bounded, and its time derivative according to the LTV equations is negatively defined, which is a criterion of stability.

New methods of LF construction for a certain class of LTV are proposed [3], Lyapunov's inverse theorem for asymptotic stability is proved. Its necessary and sufficient conditions are obtained based on the proved Lyapunov's differential inequalities [4].

With the use of Riccatti equations and matrix inequalities, an algorithm for assessing the stability of LTV, whose disturbances are described by quadratic constraints, was developed [5].

Obtaining the specified technical indicators of the LTV by using the stability theory is shown on the examples of spacecraft orientation systems [6, 7] and control of disturbed aircraft movement in the pitch plane [8].

The effectiveness of using Lyapunov's differential inequalities for the construction of the algorithm for the calculation of CL is shown, which provides a compromise between the requirements of speed and accuracy of stabilization, the properties of the transient process are established, and the assumption of a limited range of coefficient changes is removed.

The concept of building a dynamic controller in LTV feedback, when its parameters are known only approximately, has been developed [9]. The sufficient and necessary conditions for the possibility of solving the problem in the form of matrix inequalities are obtained, based on which the parameters of the controller are determined.

In the control system of the rocket rotational movement the model parameters deviation from the time-varying nominal values can amount to ten or more percent, therefore, to increase the efficiency of using the method of frozen coefficients, an algorithm for their refinement by using the data of measuring devices on the current values of part of the state vector coordinates is proposed [10]. Algorithms for specifying LTV parameters for various types of disturbances are also described in works [11-14]. The analysis of the available sources shows that due attention is not paid to the issues of developing methodical support of applied value for the study of LTV in the available sources.

OBJECTIVE AND TASKS

The goal is to develop methodological support for constructing an algorithm for determining the equivalent stationary approximation, that is, of the transfer function (TF), which is equivalent to the LTV at the selected time interval.

The task is to show the possibility of obtaining a second-order TF, which is equivalent to the LTV on a certain trajectory section, using the example of the system for controlling the rotational motion of a rocket in one plane.

MATERIALS AND METHODS

The mathematical model of LTV is a system of differential equations, the parameters of which have a constant and time-varying component. From the point of view such data as the duration of the transient process during disturbance compensation, stability margin, frequency response, and others TF (the ratio of Laplace transforms of the output signal to the input signal) gives enough information about dynamic characteristics of LTV on the chosen time interval. To obtain the TF, the system of differential equations is transformed according to Laplace with zero initial values.

Representation of the variable component of the parameters of the LTV model by the sum of exponential functions has advantages from the point of view of the level of complexity of the transition from differential equations to TF. This follows from the well-known properties of the Laplace transform of the time function into a function of the complex variable s, which is called the image:

$$L\{f(t)\} = \int_0^{\infty} f(t)\cdot e^{-s\cdot t}\cdot dt \quad . \tag{1}$$

When the variable components

$$\tilde{a}_{\psi\delta}(t), \tilde{a}_{\psi\psi}(t)$$

of the LTV model parameters $a_{\psi\delta}(t)$ and $a_{\psi\psi}(t)$ are approximated by the sum of, for example, six exponential functions, i.e.

$$\tilde{a}_{\psi\psi}(t) = \sum_{i=1}^{6} C_{\psi i}\cdot e^{r_{\psi i}\cdot t} \quad , \tag{2}$$

$$\tilde{a}_{\psi\delta}(t) = \sum_{i=1}^{6} C_{\delta i}\cdot e^{r_{\delta i}\cdot t} \quad , \tag{3}$$

then based on (1-3) the Laplace transform of individual components of the LTV equation according to the image delay theorem will be as follows:

$$L\{\psi\cdot\tilde{a}_{\psi\psi}(t)\} = L\{\psi\cdot\sum_{i=1}^{6} C_{\psi i}\cdot e^{r_{\psi i}\cdot t}\} = \sum_{i=1}^{6} C_{\psi i}\cdot\psi(s-r_{\psi i}) \quad ,$$

$$L\{\dot{\psi}\cdot\tilde{a}_{\psi\delta}(t)\} = \sum_{i=1}^{6} C_{\delta i}\cdot\psi(s-r_{\delta i})\cdot(s-r_{\delta i}) \quad .$$

The indicators $r_{\psi i}$, $r_{\delta i}$ in (2, 3) are determined by a well-known algorithm as the roots of the characteristic equation

$$r^6 + \sum_{j=1}^{6} a_j\cdot r^{6-j} = 0 \quad . \tag{4}$$

The coefficients in relation (4) are found by solving the system

$$\mathbf{mr}\cdot\begin{bmatrix}\mathbf{C}\\ \mathbf{A}\end{bmatrix} = \mathbf{G} \quad , \tag{5}$$

$$\mathbf{mr} = \begin{bmatrix} p_1(t_1)\ldots p_6(t_1) - I_1(t_1) - \ldots - I_6(t_1) \\ \ldots \\ p_1(t_{12})\ldots p_6(t_{12}) - I_1(t_{12}) - \ldots - I_6(t_{12}) \end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix} c_1 \\ \dots \\ c_6 \end{bmatrix}, \mathbf{A}=\begin{bmatrix} a_1 \\ \dots \\ a_6 \end{bmatrix}, \mathbf{G}=\begin{bmatrix} f(t_1)-f(0) \\ \dots \\ f(t_{12})-f(0) \end{bmatrix},$$

$$p_j(t)=t^j / j!, I_1(t)=\int_0^t f(\tau)\cdot d\tau, I_j(t)=\int_0^t I_{j-1}(\tau)\cdot d\tau,$$

f(t) – the variable component of the model parameter, $t_1 \dots t_{12}$ – points on the selected time interval of LTV operation.

The solution of system (4) can be obtained, including for cases of rank **mr** less than 12, by the lsolve(mr,G) procedure, which uses the LU decomposition method.

The coefficients of exponential functions, for example, in equation (2) are determined from the system:

$$\mathbf{C_\psi}^T \cdot \begin{bmatrix} e^{r_{\psi 1}\cdot\tau_i} \\ \dots \\ e^{r_{\psi 6}\cdot\tau_i} \end{bmatrix} = \tilde{a}_{\psi\psi}(\tau_i), i=\overline{1,6}$$

The accuracy estimation of approximation by exponential functions is given in the works of the Latvian authors Kulikov and Timohovich.

The possibility of obtaining an equivalent stationary approximation of LTV that is TF, for a certain time range will be shown on the example of a control system for the rocket rotational movement in one plane, the equation of which at the initial stage of development can be taken in the form:

$$\begin{gathered} \ddot{\psi}-\psi\cdot(\bar{a}_{\psi\psi}+\sum_{i=1}^{6}C_{\psi i}\cdot e^{r_{\psi i}\cdot t})- \\ -(k_\psi\cdot\psi+k_\psi^{'}\cdot\dot{\psi})\cdot(\bar{a}_{\psi\delta}+\sum_{i=1}^{6}C_{\delta i}\cdot e^{r_{\delta i}\cdot t})= \\ =(k_\psi\cdot\psi_g+k_\psi^{'}\cdot\dot{\psi}_g)\cdot(\bar{a}_{\psi\delta}+\sum_{i=1}^{6}C_{\delta i}\cdot e^{r_{\delta i}\cdot t})+m \end{gathered} \quad (6)$$

where

$$\psi, \psi_g, \dot{\psi}, \dot{\psi}_g, \ddot{\psi}$$

are actual and specified yaw angle, as well as their time derivatives;
m is disruptive acceleration;

$$\bar{a}_{\psi\psi}, \bar{a}_{\psi\delta}$$

are the constant components of the model parameters;

$$k_{\psi}, k_{\psi}^{'}$$

are the CL coefficients;

$$r_{\psi i}, r_{\delta i}, C_{\psi i}, C_{\delta i}$$

are the exponents and coefficients of the exponential functions of approximation of the corresponding variable component of the model parameters.

As it's known, the principle of superposition is valid for linear systems, according to which the result of the action of the input signal $\psi_g(t)$ or *m(t)* can be determined independently. To build an algorithm for determining the equivalent stationary approximation of the LTV at a certain time interval from two possible TFs

$$w_z(s) = \frac{L\{\psi(t)\}}{L\{\psi_g(t)\}} = \frac{\psi(s)}{\psi_g(s)},$$

$$w_m(s) = \frac{L\{\psi(t)\}}{L\{m(t)\}} = \frac{\psi(s)}{m(s)} \tag{7}$$

in this work is chosen TF $w_z(s)$, which is determined by Laplace transformation of equation (6) at zero initial values.

To obtain the TF, the differential equation (6) is transformed into an algebraic one with respect to the images of the actual ψ(s) and specified $\psi_g(s)$ value of the yaw angle:

$$\begin{gathered}
\psi(s)\cdot[s^2 - k_{\psi}^{'}\cdot\bar{a}_{\psi\delta}\cdot s - k_{\psi}\cdot\bar{a}_{\psi\delta} - \bar{a}_{\psi\psi} - \\
-\sum_{i=1}^{6} C_{\psi i}\cdot\frac{\psi(s-r_{\psi i})}{\psi(s)} - k_{\psi}\cdot\sum_{i=1}^{6} C_{\delta i}\cdot\frac{\psi(s-r_{\delta i})}{\psi(s)} - \\
-k_{\psi}^{'}\cdot\sum_{i=1}^{6} C_{\delta i}\cdot\frac{\psi(s-r_{\delta i})\cdot(s-r_{\delta i})}{\psi(s)}] = \\
= \psi_g(s)\cdot[\bar{a}_{\psi\delta}\cdot(k_{\psi} + k_{\psi}^{'}\cdot s) + \\
+k_{\psi}\cdot\sum_{i=1}^{6} C_{\delta i}\cdot\frac{\psi_g(s-r_{\delta i})}{\psi_g(s)} + \\
+k_{\psi}^{'}\cdot\sum_{i=1}^{6} C_{\delta i}\cdot\frac{\psi_g(s-r_{\delta i})\cdot(s-r_{\delta i})}{\psi_g(s)}] .
\end{gathered} \tag{8}$$

Equation (8) makes it possible to obtain the TF $w_z(s)$ in the form of a fractional-rational function of a complex-type argument *s*:

$$w_z(s)=\frac{\psi(s)}{\psi_g(s)}=\frac{P(s)}{Q(s)}$$

where

$$P(s)=\bar{a}_{\psi\delta}\cdot(k_\psi+k_\psi^{'}\cdot s)+k_\psi\cdot\sum_{i=1}^{6}C_{\delta i}\cdot\frac{\psi_g(s-r_{\delta i})}{\psi_g(s)}+$$
$$+k_\psi^{'}\cdot\sum_{i=1}^{6}C_{\delta i}\cdot\frac{\psi_g(s-r_{\delta i})\cdot(s-r_{\delta i})}{\psi_g(s)}, \qquad (9)$$

$$Q(s)=s^2-k_\psi^{'}\cdot\bar{a}_{\psi\delta}\cdot s-k_\psi\cdot\bar{a}_{\psi\delta}-\bar{a}_{\psi\psi}-$$
$$-\sum_{i=1}^{6}C_{\psi i}\cdot\frac{\psi(s-r_{\psi i})}{\psi(s)}-k_\psi\cdot\sum_{i=1}^{6}C_{\delta i}\cdot\frac{\psi(s-r_{\delta i})}{\psi(s)}-$$
$$-k_\psi^{'}\cdot\sum_{i=1}^{6}C_{\delta i}\cdot\frac{\psi(s-r_{\delta i})\cdot(s-r_{\delta i})}{\psi(s)}. \quad . \qquad (10)$$

Iterations are necessary to obtain the TF $w_z(s)$, since the image of the output signal ψ(s) is included in the last three terms of the equation (10) left part, which are a consequence of the time instability of the model parameters on the trajectory's selected section and considered as a disturbance in this work.

To obtain the first approximation of the image of the output signal $\psi_0(s)$ necessary for the iterations, the image of the signal at the input of the CS $\psi_g(s)$ is required, the stationary approximation of the LTV does not depend on the choice of which.

From the point of view of the complexity level of the algorithm, it can be taken as constant – single signal with accuracy up to the factor d, that is $\psi_g(t)=d\cdot 1(t)$. Then according to (1) $\psi_g(s)=d/s$.

When the disturbance is not taken into account, then in equation (8) terms with coefficients $C_{\psi i}$, $C_{\delta i}$ are assumed to be zero and the first approximation of the TF $w_z(s)$ will have the form

$$w_{z0}(s)=\frac{\psi_0(s)}{\psi_g(s)}=\frac{\bar{a}_{\psi\delta}\cdot(k_\psi+k_\psi^{'}\cdot s)}{s^2-\bar{a}_{\psi\delta}\cdot k_\psi^{'}\cdot s-k_\psi\cdot\bar{a}_{\psi\delta}-\bar{a}_{\psi\psi}}, \qquad (11)$$

and the first approximation of the output signal image

$$\psi_0(s)=\psi_g(s)\cdot w_{z0}(s)=d\cdot w_{z0}(s)/s$$

while the TF $w_z(s)$ (11) numerator

$$P(s) = \bar{a}_{\psi\delta} \cdot (k_{\psi} + k_{\psi}^{'} \cdot s) +$$
$$+ s \cdot (k_{\psi} \cdot \sum_{i=1}^{6} \frac{C_{\delta i} \cdot}{s - r_{\delta i}} + k_{\psi}^{'} \cdot \sum_{i=1}^{6} C_{\delta i}) \qquad (12)$$

according to equation (8) and the relations for the terms of its right-hand side:

$$\frac{\psi_g(s-\alpha)}{\psi_g(s)} = \frac{s}{s-\alpha}, \quad \frac{\psi_g(s-\alpha)\cdot(s-\alpha)}{\psi_g(s)} = s \, .$$

The coefficients CL and, which are included in (6, 8-12), are determined for the selected interval of the trajectory based on the given preliminary values of the stability margin η_1 on the roots plane of the characteristic polynomial (CHP) and the frequency f_1 of oscillations of the missile body in the transient process of disturbance compensation:

$$. \; k_{\psi} = -(\eta_1^2 + 4\pi^2 \cdot f_1^2 + \bar{a}_{\psi\psi}) / \bar{a}_{\psi\delta}, \quad k_{\psi}^{'} = -2\eta_1 / \bar{a}_{\psi\delta}. \qquad (13)$$

The relations (13) follow from the fact that the roots of the denominator Q_0 of the TF (11) first approximation according to the given values η1 and f1 are as follows:

$$s_{1,2} = -\eta_1 \pm j \cdot 2\pi \cdot f_1 \, ,$$

where $j^2 = -1$.

Iterations to determine the denominator $Q(s)$ TF $w_z(s)$ that is characteristic polynomial (CHP) can be carried out according to the scheme:

$$Q_k(s) = Q_0(s) - \frac{1}{\psi_{k-1}(s)} \cdot \left[\sum_{i=1}^{6} C_{\psi i} \cdot \psi_{k-1}(s - r_{\psi i}) - \right.$$
$$- k_{\psi} \cdot \sum_{i=1}^{6} C_{\delta i} \cdot \psi_{k-1}(s - r_{\delta i}) -$$
$$\left. - k_{\psi}^{'} \cdot \sum_{i=1}^{6} C_{\delta i} \cdot \psi_{k-1}(s - r_{\delta i}) \cdot (s - r_{\delta i}) \right]; \qquad (14)$$
$$w_{zk}(s) = \frac{P(s)}{Q_k(s)}; \; \psi_k(s) = \psi_g(s) \cdot w_{zk}(s) =$$
$$= w_{zk}(s) \cdot d / s; \; k = \overline{1, n} \, ,$$

where the index k is the number of the iteration step, $Q_0(s)$ is the denominator of the TF (13), in which the disturbance is not taken into account.

At each step of the iteration, an array N of l rows and two columns is created, in which the values of CHP $Q_k(s)$ are entered, where the argument s varies over the range of characteristic frequencies of the control system.

By processing this array with the use of method of least squares (l equations with three unknown coefficients of the CHP), the current coefficients q_{2k}, q_{1k}, q_{0k} of the CHP and, accordingly, the values η_{2k}, f_{2k} are determined.

The number of iteration steps n depends on the results of checking the achievement of the specified value of the difference of the modules selected to control the convergence of the values at the current and previous step, for example η_{2k} - η_{2k-1}, or f_{2k} - f_{2k-1}.

The result of the performed iterations is the indicator η_2 of the stability margin on the CHP roots plane and TF (7) of the closed system

$$w_z(s) = \frac{P(s)}{q_2 \cdot s^2 + q_1 \cdot s + q_0} \quad (15)$$

The convergence of the iterations has been checked on the example of the rotary motion control system in the yaw plane of the space rocket first stage, with the data in the table 1.

Table 1 – Data for calculation coefficients of CL

$\bar{a}_{\psi\psi}$	$\bar{a}_{\psi\delta}$	η_1	f_1
s^{-2}		s^{-1}	Hz
0.849	-0.331	1.2	0.3
		0.5	

Results

The advantage of representing the variable components of the model parameters as a sum of exponential functions is a simple transition from the control system differential equations (6) to their Laplace transformation, and the disadvantage is that iterations are necessary to obtain the TF. This can be seen from equation (8), in which the image $\psi(s)$ of the system output signal is included in the terms of the left part of the equation, which are due to the instability of the parameters.

For the selected data example, three iterations were enough so that the indicator η_2 of the stability margin of the missile rotational motion control system, taking into account the instability of the model parameters, was calculated with an error of no more than 0.01 s^{-1}.

The results of the conducted experiments show the possibility of constructing an algorithm for calculating the stationary approximation of the LTV on the selected trajectory section by obtaining the equivalent TF using the Laplace transformation of the time-varying component of the models parameter, given by the sum of exponential functions.

Conclusions

The scientific novelty of the work consists in the development of a methodology for determining the stationary approximation of the LTV by Laplace transformation of the variable component of the mathematical model parameters, represented by the sum of exponential functions.

The practical significance of the obtained results is the expansion of the methodological base for designing systems with time-varying parameters.

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