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Evaluation of dynamic characteristics of a linear time-varying system

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Purpose. The development of methodological support for the construction of an algorithm for calculating the coefficients of the transfer function of the second order link, which is equivalent to a time-varying system in the selected time interval from the point of view of the smallest average value of the modulus of the difference of dimensionless state vectors. **Design / Method / Approach**. Mathematical models of a time-varying system and a second-order link are used, along with a criterion that determines the transfer function coefficients. The Levenberg-Marquardt algorithm finds the minimum, and the Runge-Kutta algorithm solves differential equations. The output of the time-varying system is obtained numerically, while the second-order link's output is an analytical solution. **Findings**. Based on the calculations carried out for the selected data example, the possibility of determining the transfer function coefficients of the second-order link is shown, which, from the point of view of the smallest average value of the modulus of the difference of dimensionless state vectors on the selected time interval, is equivalent to time-varying system. **Theoretical Implications**. It is possible to have an estimate of the margin of stability, type and duration of the transient process during the selected time interval of the system operation by using the mathematical apparatus of linear stationary systems. **Practical Implications**. It leads to the expansion of the methodological base of analysis and synthesis of linear time-varying systems. **Originality / Value**. It lies in the using the Levenberg-Marquardt method to determine the coefficients of the transfer function which is equivalent to the equations of a time-varying system at a certain time interval from the point of view of the selected criterion. **Research Limitations / Future Research**. The algorithm was developed for the rocket rotational control system in one plane. The transfer function coefficients depend on constraints and the test signal within 15%. Further research includes an equivalent stationary approximation considering actuator inertia and center of mass disturbances. **Article Type**. Conceptual.

Keywords:

time-varying control system, transfer function, equivalence criterion, Levenberg-Marquardt algorithm

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Мета. Розробка методичного забезпечення побудови алгоритму розрахунку коефіцієнтів передатної функції ланки другого порядку, що є еквівалентною часово-змінній системі на обраному часовому інтервалі з точки зору найменшого середнього значення модуля різниці безрозмірних векторів стану. **Дизайн / Метод / Підхід.** Використовуються математичні моделі часово-змінної системи та ланки другого порядку, а також критерій для визначення коефіцієнтів передатної функції. Мінімізація здійснюється методом Левенберга-Маркуардта, розв'язання диференціальних рівнянь – методом Рунге-Кутта. Вихідний сигнал часово-змінної системи отримується чисельно, а ланки другого порядку – аналітично. **Результати**. На підставі розрахунків, проведених для обраного прикладу даних, продемонстровано можливість визначення коефіцієнтів передатної функції ланки другого порядку, яка з точки зору найменшого середнього значення модуля різниці безрозмірних векторів стану на обраному часовому інтервалі є еквівалентною часово-змінній системі. **Теоретичне значення**. За допомогою математичного апарату лінійних стаціонарних систем можливо оцінити запас стійкості, тип та тривалість перехідного процесу протягом обраного часового інтервалу експлуатації системи. **Практичне значення**. Це сприяє розширенню методичної бази аналізу та синтезу лінійних часово-змінних систем. **Оригінальність / Цінність**. Вона полягає у застосуванні методу Левенберга-Маркуардта для визначення коефіцієнтів передатної функції, що є еквівалентною рівнянням часово-змінної системи на певному часовому інтервалі з точки зору обраного критерію. **Обмеження дослідження / Майбутні дослідження**. Алгоритм розроблено для системи керування обертанням ракети в одній площині. Коефіцієнти передатної функції залежать від обмежень і тестового сигналу (до 15%). Подальші дослідження охоплюють еквівалентну стаціонарну апроксимацію з урахуванням інерційності виконавчого пристрою та збурень центру мас. **Тип статті**. Концептуальна.

Ключові слова:

нестаціонарна система управління, передатна функція, критерій еквівалентності, алгоритм Левенберга-Марквардта

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The analysis of dynamic characteristics of linear time-varying (LTV) systems compared to stationary systems is significantly more complex due to several fundamental reasons. Firstly, the classical concept of stability requires reformulation when applied to LTV systems. Unlike time-invariant systems, where stability can be directly assessed through eigenvalues of the system matrix, in LTV systems, there is no direct correlation between stability properties and the eigenvalues of the coefficient matrix of the system of equations. This complicates the derivation of general stability criteria and necessitates alternative analytical approaches.

To determine an appropriate control law that ensures the desired dynamic characteristics of an LTV system, various mathematical approaches have been explored. These include differential inequalities, parametric Lyapunov equations, predictive control models, and differential equations with constant coefficients approximated around specific time intervals. Each of these methods offers advantages and limitations depending on the class of LTV systems under consideration.

The construction of a Lyapunov function (LF) for LTV systems is particularly challenging, as it requires solving a scalar differential equation that incorporates both improper and double integrals (Zhou et al., 2020). For scalar LTV systems, an LF construction method based on the integral of system parameters with a weight function over a finite time interval has been proposed. Specific constraints are imposed on the weight function to ensure that the Lyapunov function remains positively defined and uniformly bounded. Furthermore, its time derivative, when evaluated according to the governing equations of the LTV system, must be negatively defined—fulfilling a necessary stability criterion.

Several novel methods for constructing LF for specific classes of LTV systems have been introduced (Kawano, 2020), including a proof of Lyapunov's inverse theorem for asymptotic stability. Necessary and sufficient conditions for stability have been established based on differential inequalities derived from Lyapunov's approach (Zhou, 2016). Additionally, stability assessment algorithms employing Riccati equations and matrix inequalities have been developed to handle LTV systems subjected to disturbances constrained by quadratic bounds (Seiler et al., 2019).

The application of stability theory to achieve predefined technical performance characteristics in LTV systems has been demonstrated in various practical domains. Examples include spacecraft orientation control (Zhou, 2021; Mullhaupt et al., 2007) and the regulation of perturbed aircraft motion in pitch dynamics (Xie et al., 2022). Despite these advancements, an analysis of the available literature reveals that insufficient attention has been dedicated to the development of methodological frameworks with direct applied value for LTV system analysis and synthesis.

In classical control theory, the transfer function (TF) is used to determine the dynamic characteristics of a linear sta-tionary system and is defined as the ratio of the Laplace transform of the system's output signal to the Laplace transform of its input signal. The determination of the TF coefficients for a second-order link that is equivalent to an LTV system over a finite time interval requires approximating the time-dependent coefficients of the governing differential equations. One approach involves representing these variable components as exponential functions (Avdieiev, 2024a), whose products with the system state variables and their derivatives are subsequently transformed via Laplace methods. Iterative refinement of these transformations ensures the accuracy of the resulting TF coefficients.

To further simplify the computational algorithm for determining TF coefficients, as compared to the methodology presented in Avdieiev (2024a), the present study aims to develop a methodological framework for constructing an algorithm that minimizes the average absolute deviation between the dimensionless output state vectors of the LTV system and its second-order link approximation. This optimization is performed using the Levenberg-Marquardt algorithm, which provides a robust numerical approach to achieving the desired coefficient accuracy.

Addressing this problem is particularly relevant, as the existing literature does not sufficiently cover the methodological support necessary for the practical analysis and synthesis of LTV systems. By refining the methodological foundations of transfer function approximation, this study contributes to the broader effort of improving control strategies for complex time-varying systems.

Mathematical Framework

The solution to the named task is shown on the example of a time-varying system for controlling the rotational movement of a rocket in one plane.

Without taking into account the executive device inertia, the disturbed movement of the mass center, fuel fluctuations and the body final stiffness, the system equation is as follows:

$$\dot{x} = a(t) \cdot x + f(t); \quad x = \begin{bmatrix} \psi \\ \dot{\psi} \end{bmatrix};$$

$$a(t) = \begin{bmatrix} 0 & 1 \\ q_0(t) & q_1(t) \end{bmatrix}; \tag{1}$$

$$f(t) = \begin{bmatrix} 0 \\ p_0(t) \cdot \psi_g(t) + p_1(t) \cdot \dot{\psi}_g(t) \end{bmatrix}.$$

In equation (1) ψ, $\dot{\psi}$ are the rotation angle of the missile body and its time derivative; $\psi_g(t)$ is the input signal depending on time, that means the specified value of the missile body rotation angle; $q_0(t)$, $q_1(t)$, $p_0(t)$, $p_1(t)$ are variable coefficients that depends on the rocket inertial mass and aerodynamic characteristics, altitude and flight speed.

The solution of system (1) can be obtained numerically, for example, by the Runge-Kutta method, its results are presented in a table, denote them $\psi_н(t)$ and $\dot{\psi}_н(t)$. They will be used in the iterative process of determining TF coefficients using the Levenberg-Marquardt method.

As you know, TF is the Laplace transform ratio of the output signal of the system ψ(t)

$$L\{\psi(t)\} = \int_0^\infty \psi(t) \cdot e^{-s \cdot t} \cdot dt = \psi(s)$$

to the Laplace transform of the input signal $\psi_g(s)$, i.e.

$$w(s) = \frac{L\{\psi(t)\}}{L\{\psi_g(t)\}} = \frac{\psi(s)}{\psi_g(s)},$$

where L is the designation of the Laplace transform operator, s is a variable of complex type.

To obtain the first approximation of the TF based on equation (1), the coefficients q_0, q_1, p_0, p_1 are assumed to be constant and equal, for example, to their value at the midpoint of the selected time interval. This makes it possible to transform these equations according to Laplace and determine the TF of the second-order link in the form:

$$w(s) = \frac{\psi(s)}{\psi_g(s)} = \frac{p_1 \cdot s + p_0}{s^2 - p_1 \cdot s + q_0} = \frac{p_1 \cdot s + p_0}{s^2 - 2\alpha \cdot s + \alpha^2 + \beta^2}, \tag{2}$$

where α, β are the real and imaginary part of the roots of the equation $s^2 - p_1 \cdot s + q_0 = 0$.

As can be seen from (2), the search for a TF equivalent to a time-varying system should be carried out in the four-dimensional space of coefficients α, β, p_0, p_1.

To reduce the duration of the iterative process of finding the TF coefficients of the second-order equivalent link the solution of its differential equation

$$\ddot{\psi} - 2\alpha \cdot \dot{\psi} + (\alpha^2 + \beta^2) \cdot \psi =$$

$$= p_1 \cdot \dot{\psi}_g(t) + p_0 \cdot \psi_g(t), \tag{3}$$

which follows from the TF (2), the initial conditions and the input signal $\psi_g(t)$, must be obtained analytically.

In this work, two variants of input signals are considered:

– in the form of the parabola equation with a vertex in the center of a certain time interval, for example $0 \ldots 2t_p$, and equal to zero at its edges

$$\psi_{g1}(t) = a_1 \cdot t^2 + b_1 \cdot t + c_1; \tag{4}$$

– in the form of two equations that specify the program for turning the rocket body to a given angle during the $2t_p$ time interval

$$\psi_{g2}(t) =$$

$$= \begin{cases} f_i(t) = em \cdot \left(\frac{t^2}{2} + \frac{t}{d} - \frac{e^{d \cdot t}}{d^2}\right), \quad 0 \le t \le t_p \\ f_i(t_p) + em \cdot \left(t + \frac{1 - e^{d \cdot t}}{d^2}\right) \cdot (t - t_p) - f_i(t - t_p), \quad t_p < t \le 2t_p \end{cases} \tag{5}$$

where the coefficients depend on the given value of the parabola at the top in the center of the interval, the magnitude of the angle of rotation and the desired transition process profile.

For the case of zero initial values and the signal ψ_{g1} (4) at the input of the system, the solution of equation (3)

$$\psi(\alpha, \beta, p_0, p_1, t) =$$
$$= e^{\alpha t} \cdot (A_c \cdot \cos\beta\, t + A_s \cdot \sin\beta\, t) +$$
$$+ A \cdot t^2 + B \cdot t + C, \tag{6}$$

where

$$A = \frac{p_0 \cdot a_1}{\alpha^2 + \beta^2},$$
$$B = \frac{1}{\alpha^2 + \beta^2} \cdot (2a_1 \cdot p_1 + p_0 \cdot b_1 + 4A \cdot \alpha),$$
$$C = \frac{1}{\alpha^2 + \beta^2} \cdot (a_1 \cdot p_1 + p_0 \cdot c_1 - 2A + 2\alpha \cdot B),$$
$$A_c = -C,$$
$$A_s = \frac{-(B + \alpha \cdot A_c)}{\beta}.$$

For the case of zero initial values and signal ψ_{g2} (5) at the input of the system, the solution of equation (3) can be written in a form like (6), but with other coefficients in its terms.

In the first half of a certain trajectory section

$$\psi(\alpha, \beta, p_0, p_1, t) = e^{\alpha t} \cdot$$
$$\cdot (A_c \cdot \cos\beta\, t + A_s \cdot \sin\beta\, t) +$$
$$+ A \cdot t^2 + B \cdot t + C + f_e \cdot e^{d \cdot t},$$
$$0 \le t \le t_p, \tag{7}$$

where

$$A = \frac{p_0 \cdot em}{2 \cdot (\alpha^2 + \beta^2)},$$
$$B = \frac{em}{\alpha^2 + \beta^2} \cdot \left(p_1 + \frac{p_0}{d} + 4A \cdot \alpha\right),$$
$$C = \frac{1}{\alpha^2 + \beta^2} \cdot \left(\frac{em \cdot p_1}{d} - 2\alpha \cdot B\right),$$
$$f_e = \frac{-em \cdot \left(\frac{p_0}{d} + p_1\right)}{d \cdot (d^2 - 2\alpha \cdot d + \alpha^2 + \beta^2)},$$
$$A_c = -C - f_e,$$
$$A_s = \frac{-d \cdot f_e - B - A_c}{\beta}.$$

In the second half of a certain trajectory section

$$\psi(\alpha, \beta, p_0, p_1, t) =$$
$$= e^{\alpha \cdot (t - t_p)} \cdot [A_c \cdot \cos\beta\,(t - t_p) +$$
$$+ A_s \cdot \sin\beta\,(t - t_p)) + A \cdot (t - t_p)^2 +$$
$$+ B \cdot (t - t_p) + C + f_e \cdot e^{d \cdot (t - t_p)}, \tag{8}$$
$$t_p < t \le 2t_p$$

where

$$A = \frac{-em \cdot p_0}{2 \cdot (\alpha^2 + \beta^2)},$$
$$B = \frac{1}{\alpha^2 + \beta^2} \left[f_a(t_p) \cdot p_0 - em \cdot \left(\frac{p_0}{d} + p_1\right) + 2\alpha \cdot A\right],$$
$$C = \frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} p_0 \cdot f_i(t_p) + \\ + p_1 \cdot f_a(t_p) - \frac{p_1 \cdot em}{d} - 2A + 2\alpha \cdot B \end{bmatrix},$$
$$f_e = \frac{em \cdot \left(p_1 + \frac{p_0}{d}\right)}{d \cdot (d^2 - 2\alpha \cdot d + \alpha^2 + \beta^2)},$$
$$A_c = \psi(t_p) - C - f_e,$$
$$A_s = \frac{1}{\beta}\left[\dot\psi(t_p) - \alpha \cdot A_c - f_e \cdot d\right],$$
$$f_a(t) = em \cdot \left(t + \frac{1 - e^{d \cdot t}}{d}\right).$$

The equivalence of a link with a TF of type (2) to a time-varying system is established by comparing their transient processes at the selected test signals ψ_{g1}, ψ_{g2} (4, 5). The coefficients of the link are found by minimizing the criterion, the value of which quantitatively characterizes the results of the transient processes comparison.

The work considers variants of criteria from the point of view of local extrema. Part of them is described in (Avdieiev, 2024b).

Criterion of the minimum sum of the squares of the difference in the angles of missile body rotation

$$Q_1(r) = \sum_{i=1}^{n} [\psi(r, t_i) - \psi_{н}(t_i)]^2, \tag{9}$$
$$t_i = i \cdot \Delta t,$$
$$n = \frac{2t_p}{\Delta t},$$

where Δt is the integration step of the system of equations (1),

$$r = [\alpha, \beta, p_0, p_1]^T$$

is the vector of TF coefficients introduced to shorten the record.

Criterion of the minimum square of the difference of the angles of the state vector direction

$$Q_2(r) = \sum_{i=1}^{n} \begin{bmatrix} arctg\left[\frac{\dot\psi(r, t_i)}{\psi(r, t_i)}\right] - \\ - arctg\left[\frac{\dot\psi_{н}(t_i)}{\psi_{н}(t_i)}\right] \end{bmatrix}^2. \tag{10}$$

Criterion of the minimum square of the difference of the modules of the dimensionless state vector

$$Q_3(r) = \sum_{i=1}^{n} [m(r, t_i) - m_{н}(t_i)]^2, \tag{11}$$

where

$$m(r, t) = \sqrt{[\psi(r, t)/\psi_m]^2 + [\dot\psi(r, t)/\psi_{mt}]^2},$$
$$m_{н}(t) = \sqrt{[\psi_{н}(t)/\psi_m]^2 + [\dot\psi_{н}(t)/\psi_{mt}]^2};$$

ψ_m, ψ_{mt} are the angle and angular speed of the body rotation, used for the transition to dimensionless quantities.

Criterion of the minimum value of the area difference under the curves of transient processes

$$Q_4(r) = \Delta t \cdot \left|\sum_{i=1}^{n} [\psi(r, t_i) - \psi_{н}(t_i)]\right|. \tag{12}$$

The criterion for the minimum of the average value of the difference between the state vectors

$$Q_5(r) = \frac{1}{n} \sum_{i=1}^{n} \sqrt{\left[\frac{[\psi(r, t_i) - \psi_{н}(t_i)]}{\psi_m]^2} + \frac{[\dot\psi(r, t_i) - \dot\psi_{н}(t_i)]}{\psi_{mt}]^2}\right]}. \tag{13}$$

The criterion of the minimum value of the difference in the area of phase portraits

$$Q_6(r) = |S_f(r) - S_{н}|, \tag{14}$$

where

$$S_f(r) = \int_0^{\psi_{max}} \dot\psi(r, t) \cdot d\psi(r, t) =$$
$$= \int_0^{2t_p} \dot\psi(r, t)^2 \cdot dt = \Delta t \cdot \sum_{i=1}^{n} \dot\psi(r, t_i)^2,$$
$$S_H = \Delta t \cdot \sum_{i=1}^{n} \psi_n(t_i)^2$$

From the point of view of the local extrema location, the properties of criteria (9-14) were established experimentally using the Levenberg-Marquardt method on the example of the data in the table 1 with test signals ψg_1, ψg_2 (4, 5).

As the conducted experiments have shown, after studying the results of using a certain criterion for the purpose of quantitative analysis of the equivalence level of equations (1) and TF (2) depending on time, it may be appropriate to construct a difference phase portrait in coordinates

$$\psi(r) - \psi_{\text{н}}\ \dot{\psi}(r) - \dot{\psi}_{\text{н}}.$$

Table 1 – dependence of TF coefficients on time (developed by author)

t	q_0	p_0	p_1
s	s^{-2}		s^{-1}
0	3.31	-4.53	-1.87
16	4.56	-5.52	-2.28
32	7.14	-7.51	-3.10

Based on the location of local extrema in the space of TF coefficients, that is, coordinates of vector **r,** and the minimum values of the criterion, option Q_5 (13) is preferred. The results of its use are in the table 2.

Table 2 – TF coefficients before and after minimizing the criterion Q_5(r) (developed by author)

Test signal	Definition point	α	β	p_0	p_1	Q_5
ψ_{g1}	primary	– 1.20	1.88	– 5.85	– 2.4	0.075
	final	– 1.23	0.57	– 2.20	– 3.7	0.046
ψ_{g2}	primary	– 1.20	1.88	– 5.85	– 2.4	0.340
	final	– 1.02	0.63	– 1.60	– 3.7	0.091

Based on the final results of determining the TF coefficients (2) in the case of the test signal ψ_{g1} (table 2), we will obtain an estimate of the following dynamic characteristics of the time-varying system in the relative time range 0...32 *s* (table 1): stability margin on the roots plane of the characteristic polynomial 1.2 s^{-1}, the duration of the transient process is 2.4 *s*, the frequency of the oscillatory component of the transient process is 0.09 Hz, the amplitude-phase dependence on the circular frequency ω

$$w(j\omega) = \frac{\psi(j\omega)}{\psi_g(j\omega)} = \frac{-4-11.5\omega^2 + j\omega\cdot(1.3+3.7\omega^2)}{\omega^4 - 3.7\omega^2 + 17},$$

where $j^2 = -1$.

The named estimates of dynamic characteristics can be used to make technical decisions in the process of designing systems with time-varying parameters.

Conclusions

Based on the calculations performed for the selected data example, this study demonstrates the feasibility of determining the transfer function (TF) coefficients for a second-order link. From the perspective of minimizing the average absolute deviation between the dimensionless state vectors over a selected time interval, the obtained TF is equivalent to the given linear time-varying (LTV) system.

The application of the transfer function approach enables the estimation of key dynamic characteristics, including the stability margin in the root plane of the characteristic polynomial, the nature and duration of transient processes, as well as the system's gain as a function of input signal frequency. This allows for a comprehensive analysis of amplitude-frequency and phase-frequency dependencies, which are crucial for assessing system behavior under various operating conditions.

A novel aspect of this study is the use of the Levenberg-Marquardt method to determine the TF coefficients, ensuring their equivalence to the governing equations of an LTV system over a specified time interval based on the selected optimization criterion. This approach enhances the accuracy of TF-based approximations and provides a refined tool for analyzing time-varying dynamic systems.

The practical significance of this work lies in expanding the methodological framework for the analysis and synthesis of LTV systems. The proposed methodology offers a structured approach for approximating LTV system dynamics using TF representations, contributing to the development of more effective modeling and control strategies.

A potential direction for future research is the determination of an equivalent stationary approximation for the LTV system governing the rocket's rotational motion. This would involve accounting for the inertia of the actuator and the effects of disturbed motion of the center of mass, further refining the accuracy of system modeling and control.

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